Some calculations in Analysis (Wallis & Stirling formulas).

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1. Wallis formula.

Let $I_n := \int_0^{\pi/2} \sin^n x dx, n \in \mathbb{N} \cup \{0\}$. Follow to definition we have $I_0 = \dfrac{\pi}{2}$,
$I_1 = \int_0^{\pi/2} \sin x dx = (-\cos x)_0^{\pi/2} = 1$. Note that for any $x \in [0, \pi/2]$ holds
$0 \le \sin x \le 1 \Rightarrow \sin^{n+1} x \le \sin^n x \le \sin^{n-1} x \Rightarrow$

(**1**) $\quad I_{n+1} \le I_n \le I_{n-1}, n \in \mathbb{N}.$

Also note that $I_n = \int_0^{\pi/2} \sin^{n-2} x(1-\cos^2 x)dx = I_{n-2} - \int_0^{\pi/2} \sin^{n-2} x \cos^2 x dx$, where

$$\int_0^{\pi/2} \sin^{n-2} x \cos^2 x dx = \left[\begin{array}{c} u' = \sin^{n-2} x \cos x;\ u = \dfrac{\sin^{n-1} x}{n-1} \\ v = \cos x;\ v' = -\sin x \end{array}\right] =$$

$\left(\dfrac{\sin^{n-1} x}{n-1} \cdot \cos x\right)_0^{\pi/2} + \int_0^{\pi/2} \dfrac{\sin^n x}{n-1} dx = \dfrac{1}{n-1} I_n$. Hence, $I_n = I_{n-2} - \dfrac{1}{n-1} I_n \Leftrightarrow$

(**2**) $\quad I_n = \dfrac{n-1}{n} I_{n-2}, n \ge 2.$

In particular we have $\quad I_{2n} = \dfrac{2n-1}{2n} I_{2(n-1)}, n \in \mathbb{N}.$

Since $I_{2n} = \dfrac{2n-1}{2n} I_{2(n-1)} \Leftrightarrow \dfrac{I_{2n}}{I_{2(n-1)}} = \dfrac{2n-1}{2n} \Rightarrow \prod_{k=1}^{n} \dfrac{I_{2k}}{I_{2(k-1)}} = \prod_{k=1}^{n} \dfrac{2k-1}{2k} \Leftrightarrow$
$\dfrac{I_{2n}}{I_0} = \dfrac{(2n-1)!!}{(2n)!!} \Leftrightarrow I_{2n} = \dfrac{(2n-1)!!}{(2n)!!} \cdot \dfrac{\pi}{2}, n \in \mathbb{N} \cup \{0\}.$

Also, $I_{2n+1} = \dfrac{2n}{2n+1} I_{2n-1} \Rightarrow \dfrac{I_{2n+1}}{I_1} = \dfrac{(2n)!!}{(2n+1)!!} \Leftrightarrow I_{2n+1} = \dfrac{(2n)!!}{(2n+1)!!}$.

Then by replacing n in inequality (**1**) with $2n$ we obtain

$$\frac{(2n)!!}{(2n+1)!!} \le \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2} \le \frac{(2n-2)!!}{(2n-1)!!} \Leftrightarrow \frac{((2n)!!)^2}{(2n+1)!!(2n-1)!!} \le \frac{\pi}{2} \le \frac{(2n)!!(2n-2)!!}{((2n-1)!!)^2} \Leftrightarrow$$

$$\frac{((2n)!!)^2(2n-1)!!}{(2n+1)!!((2n-1)!!)^2} \le \frac{\pi}{2} \le \frac{((2n)!!)^2(2n-2)!!}{((2n-1)!!)^2(2n)!!} \Leftrightarrow$$

$$\frac{((2n)!!)^2}{((2n-1)!!)^2(2n+1)} \le \frac{\pi}{2} \le \frac{((2n)!!)^2}{((2n-1)!!)^2 \cdot 2n} \Leftrightarrow$$

(**3**) $a_n \le \dfrac{\pi}{2} \le a_n \dfrac{2n+1}{2n}$, where $a_n := \dfrac{((2n)!!)^2}{((2n-1)!!)^2(2n+1)} = \prod_{k=1}^{n} \dfrac{4k^2}{4k^2-1}$.

Since $a_n \uparrow \mathbb{N}$ and $a_n \le \dfrac{\pi}{2}$ then there is $a := \lim_{n\to\infty} a_n \le \dfrac{\pi}{2}$ and, therefore,

$$\lim_{n\to\infty}\left(a_n \cdot \frac{2n+1}{2n}\right) = a.$$

Hence $\dfrac{\pi}{2} = \lim_{n\to\infty} \prod_{k=1}^{n} \dfrac{4k^2}{4k^2-1}$ that is $\dfrac{\pi}{2} = \lim_{n\to\infty} \dfrac{((2n)!!)^2}{((2n-1)!!)^2(2n+1)}$ (Wallis Formula).

2. Stirling Formula.

Lemma (extracted from [1])

There is positive constant a such that for any $n \in \mathbb{N}$ holds inequality

(1) $\quad \left(\frac{n}{e}\right)^n \sqrt{an} < n! < \left(\frac{n}{e}\right)^n \sqrt{an} \cdot e^{\frac{1}{12n}}$.

Proof.

1. First we will prove inequality $e < \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}$ for any $n \in \mathbb{N}$.

Note that sequence $\left(\left(1+\frac{2}{n-1}\right)^n\right)_{n\geq 2}$ is decreasing.

Indeed, $\left(1+\frac{2}{n-1}\right)^n > \left(1+\frac{2}{n}\right)^{n+1} \Leftrightarrow$

$\left(\frac{n+1}{n-1}\right)^n > \left(\frac{n+2}{n}\right)^{n+1} \Leftrightarrow \left(\frac{n(n+1)}{(n-1)(n+2)}\right)^n > 1+\frac{2}{n} \Leftrightarrow$

$$\left(1+\frac{2}{(n-1)(n+2)}\right)^n > 1+\frac{2}{n}.$$

Applying inequality $(1+a)^n \geq 1+na+\frac{n(n-1)}{2}a^2, a>0, n\in\mathbb{N}$ for $a = \frac{2}{(n-1)(n+2)}$

we obtain $\left(1+\frac{2}{(n-1)(n+2)}\right)^n \geq 1+\frac{2n}{(n-1)(n+2)}+\frac{n(n-1)}{2}\cdot\frac{4}{(n-1)^2(n+2)^2} =$

$1+\frac{2n}{(n-1)(n+2)}+\frac{2n}{(n-1)(n+2)^2}$ and $1+\frac{2n}{(n-1)(n+2)}+\frac{2n}{(n-1)(n+2)^2} > 1+\frac{2}{n} \Leftrightarrow$

$(n+3)n^2 > (n-1)(n+2)^2 \Leftrightarrow n^3+3n^2 > n^3+3n^2-4 \Leftrightarrow 4>0.$ ■

Since $\left(1+\frac{2}{n-1}\right)^n > \left(1+\frac{2}{n}\right)^{n+1} > \left(1+\frac{2}{n+1}\right)^{n+2}$ then

(2) $\quad \left(1+\frac{2}{n-1}\right)^n > \left(1+\frac{2}{n+1}\right)^{n+2}$

By replacing n in (**2**) with $2n+1$ we obtain $\left(1+\frac{1}{n}\right)^{2n+1} > \left(1+\frac{1}{(n+1)}\right)^{2n+3} \Leftrightarrow$

$\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} > \left(1+\frac{1}{n+1}\right)^{n+1+\frac{1}{2}}$ and, since $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} = e$ then

$$e < \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}, n \in \mathbb{N}.$$

2. Consider Taylor Representation for function $\ln\frac{1+x}{1-x}$ for $x \in (0,1)$:

$\ln\frac{1+x}{1-x} = \ln(1+x)-\ln(1-x) = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}x^k}{k} - \sum_{k=1}^{\infty}\frac{(-1)^{k-1}(-x)^k}{k} =$

$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}x^k}{k}+\sum_{k=1}^{\infty}\frac{x^k}{k} = \sum_{k=1}^{\infty}\frac{\left((-1)^{k-1}+1\right)x^k}{k} = \sum_{k=1}^{\infty}\frac{2x^{2k-1}}{2k-1}.$

Since $\frac{n+1}{n} = \frac{1+\frac{1}{2n+1}}{1-\frac{1}{2n+1}}$ then by replacing x in $\ln\frac{1+x}{1-x} = \sum_{k=1}^{\infty}\frac{2x^{2k-1}}{2k-1}$ with $\frac{1}{2n+1}$

we obtain

$\ln\frac{n+1}{n} = \sum_{k=1}^{\infty}\frac{2}{2k-1}\cdot\frac{1}{(2n+1)^{2k-1}} = \frac{2}{2n+1}\left(1+\sum_{k=2}^{\infty}\frac{1}{(2k-1)(2n+1)^{2(k-1)}}\right) <$

$\frac{2}{2n+1}\left(1+\sum_{k=1}^{\infty}\frac{1}{3(2n+1)^{2k}}\right) = \frac{2}{2n+1}\left(1+\frac{1}{3}\cdot\frac{\frac{1}{(2n+1)^2}}{1-\frac{1}{(2n+1)^2}}\right) =$

$$\frac{1}{n+\frac{1}{2}}\left(1+\frac{1}{12n(n+1)}\right) \Rightarrow \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} < e^{1+\frac{1}{12n(n+1)}}.$$

Thus we double inequality $e < \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} < e^{1+\frac{1}{12n(n+1)}}$.

Since $e < \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} \Leftrightarrow e < \frac{(n+1)^{n+\frac{1}{2}}}{n^{n+\frac{1}{2}}} \Leftrightarrow e(n+1) < \frac{(n+1)^{n+1+\frac{1}{2}}}{n^{n+\frac{1}{2}}} \Leftrightarrow$

$\frac{e^{n+1}(n+1)!}{e^n n!} < \frac{(n+1)^{n+1+\frac{1}{2}}}{n^{n+\frac{1}{2}}} \Leftrightarrow \frac{n^{n+\frac{1}{2}}}{e^n n!} < \frac{(n+1)^{(n+1)+\frac{1}{2}}}{e^{n+1}(n+1)!}, n \geq 1$ then sequence $(a_n)_{n\geq 1}$

where $a_n := \frac{n^{n+\frac{1}{2}}}{e^n n!}$, is increasing. Since $\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} < e^{1+\frac{1}{12n(n+1)}} \Leftrightarrow$

$\frac{(n+1)^{n+1+\frac{1}{2}}}{n^{n+\frac{1}{2}}} < \frac{(n+1)!}{n!} \cdot e^{n+1-n+\frac{1}{12n}-\frac{1}{12(n+1)}} \Leftrightarrow \frac{(n+1)^{n+1+\frac{1}{2}}}{(n+1)!e^{n+1-\frac{1}{12(n+1)}}} < \frac{n^{n+\frac{1}{2}}}{n!e^{n-\frac{1}{12n}}}$

then sequence $(b_n)_{n\geq 1}$, where $b_n := \frac{n^{n+\frac{1}{2}} e^{\frac{1}{12n}}}{n!e^n} = a_n e^{\frac{1}{12n}}$ is decreasing.

Since $e^{-1} = a_1 \leq a_n < b_n \leq b_1 = e^{-\frac{11}{12}}$ and $b_n = a_n e^{\frac{1}{12n}}$ then both sequences converge to the same limit.

Let $\frac{1}{a} := \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$ then $e^{\frac{11}{12}} < a < e$ and $a_n < \frac{1}{a} < b_n \Leftrightarrow \frac{1}{b_n} < a < \frac{1}{a_n} \Leftrightarrow$

$\frac{n!e^n}{n^{n+\frac{1}{2}} e^{\frac{1}{12n}}} < a < \frac{e^n n!}{n^{n+\frac{1}{2}}} \Leftrightarrow \left(\frac{n}{e}\right)^n \sqrt{an} < n! < \left(\frac{n}{e}\right)^n \sqrt{an} \cdot e^{\frac{1}{12n}} \Leftrightarrow$

(3) $\quad n! = \left(\frac{n}{e}\right)^n \sqrt{an} \cdot e^{\frac{\theta_n}{12n}}$, where $\theta_n \in (0,1)$

Using Wallis formula we can obtain $a = 2\pi$.

Indeed, since by Wallis formula $\frac{\pi}{2} = \lim_{n\to\infty} \frac{((2n)!!)^2}{((2n-1)!!)^2(2n+1)}$ and

$$\frac{(2n)!!}{(2n-1)!!} = \frac{((2n)!!)^2}{(2n)!} = \frac{2^{2n}(n!)^2}{(2n)!} = \frac{2^{2n}\left(\left(\frac{n}{e}\right)^n \sqrt{an} \cdot e^{\frac{\theta_n}{12n}}\right)^2}{\left(\frac{2n}{e}\right)^{2n} \sqrt{a \cdot 2n} \cdot e^{\frac{\theta_{2n}}{24n}}} =$$

$$\frac{2^{2n}\left(\frac{n}{e}\right)^{2n} an \cdot e^{\frac{\theta_n}{6n}}}{\left(\frac{2n}{e}\right)^{2n} \sqrt{a \cdot 2n} \cdot e^{\frac{\theta_{2n}}{24n}}} = \frac{\sqrt{an}\, e^{\frac{\theta_n}{6n}-\frac{\theta_{2n}}{24n}}}{\sqrt{2}}$$

then $\frac{\pi}{2} = \lim_{n\to\infty} \frac{an\, e^{\frac{\theta_n}{3n}-\frac{\theta_{2n}}{12n}}}{2\,(2n+1)} = \frac{a}{4} \Leftrightarrow a = 2\pi$.

So, $n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \cdot e^{\frac{\theta_n}{12n}}$, that is $\lim_{n\to\infty} \frac{n!}{\left(\frac{n}{e}\right)^n \sqrt{2\pi n}} = 1$.

1. Mathematical Reflection n.1, **2011**, **Problem U181**,
Solution by Arkady Alt, **pp.16-21**.